Development of a coupled plasticity–damage model for the study of the short and long term behaviour of confined concrete

Theoretical model and experimental results

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Notation

**Uppercase Latin**

$B^0$  Initial configuration  
$B^t$  Current configuration  
$E$  Stiffness modulus  
$E$  Stiffness modulus tensor  
$F$  Force  
$J$  Compliance function  
$K$  Unloading–reloading stiffness  
$L$  Length of a member  
$R$  Relaxation function  
$X$  Initial coordinates of a point

**Lowercase Latin**

$d$  Damage parameter  
$f_c$  Uniaxial compressive resistance  
$f_{cc}$  Multiaxial compressive resistance  
$f_{ct}$  Uniaxial tensile resistance  
$f_{ctt}$  Multiaxial tensile resistance  
$p$  Crack closure parameter  
$s_c$  Crack spacing  
$u$  Displacements of a given point  
$w$  Crack opening  
$w_p$  Plastic crack opening  
$x$  Current coordinates of a point

**Uppercase Greek**

$\Delta$  Increment  
$\Lambda$  Plastic strains coefficient

**Lowercase Greek**

$\chi$  Aging coefficient
Notation

\( \varepsilon \) Strain
\( \varepsilon_{an} \) Anelastic strain
\( \varepsilon \) Strain tensor
\( \varepsilon_e \) Elastic strain
\( \varepsilon_e \) Elastic strain tensor
\( \varepsilon_N \) Cyclic loading strain
\( \varepsilon_p \) Plastic strain
\( \varepsilon_{p,c} \) Plastic strain under compression
\( \varepsilon_{p,t} \) Plastic strain under tension
\( \varepsilon_{lat} \) Lateral strain
\( \varepsilon_3 \) Longitudinal strain
\( \varepsilon_p \) Plastic strain tensor
\( \varepsilon_v \) Viscous strain
\( \phi_t \) Function defining the movement of a point
\( \varphi \) Creep coefficient
\( \psi \) Softening function
\( \sigma \) Stress
\( \sigma \) Stress tensor
\( \tilde{\sigma} \) Nominal stress
\( \tilde{\sigma} \) Nominal stress tensor
\( \sigma_{lat} \) Lateral pressure
\( \sigma_3 \) Longitudinal stress
Chapter 1

Introduction

1.1 Problem definition, historical note

The confined response of concrete can be seen as a particular case of its three–dimensional behaviour when two of its principal stresses have the same value. This condition modifies its uniaxial (unconfined) response increasing both the strength and ductility of the material. Many structural problems have to take into account the confined response of concrete, mainly pieces with transverse reinforcement (columns with stirrups or encased in steel tubes) or confined struts in D regions (necessary in the detailing of structures and to understand several phenomena like for instance the bond behaviour). Due to this reason, wide research has always been performed on this topic, comprising experimental tests and theoretical models.

The first experimental studies on the confined behaviour of concrete were performed at the beginning of the past century. According to Neville [42], Woolson reported in 1905 the creep of concrete under high stresses when it was encased in a steel tube.

However, the first experimental campaign that was seriously performed on this topic was not developed until 1928, when Richart [49] tested various specimens of different strengths and proposed a Mohr–Coulomb criterion in order to predict the ultimate load of confined concrete. Concerned of the importance of the problem, other researchers like for instance Leon [35] studied and proposed different failure criteria for the confined concrete. Also, Kupfer performed in 1973 [30] a deep study of the failure of concrete under a triaxial state of stresses (completed with several biaxial tests) defining a general failure surface for it. A similar research has been recently performed by Curbach, Hampel et alli [12] completing the experimental results of Kupfer for high strength concrete.

In the past years, many research has been performed focusing on the confined response of plain concrete as well as its applications to other problems.

Concerning the confined response of concrete, two approaches have been dominant for its modeling. One, more general, with a three–dimensional study of the problem based on the continuum mechanics theory and other, more specific, based on the development of analytical laws valid only for confined cases.

In the first group, interesting contributions have been performed to the three–dimensional modeling of concrete introducing coupled plasticity–damage models (Lee [33], Nechnech [41], Sfer [55], …) as well as approaches based on the lattice model (see for instance Cusatis, Bažant and Cedolin [13]).

In the second group, the development of analytical laws for confined concrete, the studies of Pantazopoulou and Mills (1995, [43]) and Imran and Pantazopoulou (1996, [25]) clearly set a
depart point for further studies. Other interesting researches carried in the past years in this line have been for instance those performed by Candappa et alii [7] or Lokuge et alli [36]. Finally, several problems where the concrete is in a confined state have been widely studied for certain structural elements. For instance, reinforced concrete columns with stirrups have been studied by Sigrist [56], Légeron and Paultre [34] and Cavalcanti et alii [8] among others, and steel encased concrete columns by Lahlou and Lachemi [32], Becque et alii [6] and Fam et alii [16].

1.2 Aims and scope of the work

This study presents the application of a coupled plasticity–damage model to reproduce the mechanical behaviour of confined concrete. The model focuses on certain aspects, mainly:

- Short–term confined response of the material under compression and tension as well as the unloading–reloading behaviour when a lateral pressure is applied to it
- Long–term response of the concrete under combined cyclic and maintained loading
- Definition of a failure criterion for short and long term loading
- Definition of a damage parameter considering the load history of the material

The model has been applied to several problems in order to reproduce their response under different loading conditions, for instance:

- Confined and unconfined plain concrete specimens, comprising lateral compression and tension
- Fibre reinforced concrete, introducing the effect of fibres as an equivalent confinement pressure
- Steel encased concrete columns, applying again an equivalent confinement pressure in order to obtain their response

The applicability of this model is not only restraint to these cases and also other problems like for instance the bond phenomenon in reinforced concrete and the behaviour of confined struts in D regions can be studied applying these ideas; however, they are beyond the scope of this work.

1.3 Methodology and organisation

The study begins reviewing the approach to the problem based on the continuum mechanics principles and developing an isotropic coupled damage–plasticity model. Once the three–dimensional case has been studied, the one–dimensional application of the model is detailed. Later, an analytical characterization of the short–term response of concrete when a lateral pressure is applied to it is performed. This characterization comprises all the possible tension and compression cases as well as the unloading and reloading behaviour. Also, the results obtained with the model are compared with several tests carried out under different loading conditions.

The next two chapters are devoted to the long–term behaviour of concrete, focusing on:
• Maintained loading with its non-linear behaviour under high pressures as well as the effect of static fatigue (failure under maintained load)

• Cyclic loading with its effect on the delayed strains of concrete as well as the fatigue phenomenon
Chapter 2

Isotropic continuum models

2.1 3–D setting of the model

In this section, the continuum mechanics principles will be reviewed and used in order to
develop a three–dimensional isotropic model of behaviour where a coupled plasticity–damage
response can be introduced.

The state of a body will be defined for a given time by its stress tensor \( \sigma \), and its strain
tensor \( \varepsilon \), obtaining the latter from the displacement field as follows:

\[
\varepsilon = \frac{1}{2} (\nabla u + \nabla^T u)
\]

(2.1)

The displacement field \( u \) is in his turn obtained from the deformation \( \phi_t \) of the body
using the following relationships:

\[
x = \phi_t(X)
\]

(2.2)

\[
u = \phi_t(X) - X
\]

(2.3)

where \( x \) defines the current configuration and \( X \) the initial configuration as can be seen in
figure 2.1.

![Figure 2.1: Initial and current configurations](image)

2.1.1 Isotropic damage model

The constitutive relationship for a isotropic damage model may be adopted as follows:
\( \sigma = E : \varepsilon \) \hfill (2.4)
\[ E = E_0(1 - d) \] \hfill (2.5)

where \( d \) is the damage parameter of the body (\( 0 \leq d \leq 1 \)). With it, the nominal stress of the body can be defined as follows:
\[ \bar{\sigma} = \frac{1}{1 - d} \cdot \sigma \Rightarrow \bar{\sigma} = E_0 : \varepsilon \] \hfill (2.6)

### 2.1.2 Plasticity model

In the case of a plasticity model, the relationship between the stresses and the strains in a body are given as follows:
\[ \varepsilon = \varepsilon_e + \varepsilon_p \] \hfill (2.7)
\[ F(\sigma, \varepsilon_p) \leq 0 \] \hfill (2.8)

where \( F \) is a function of the different components of the stress tensor as well as of the developed plastic strains. The increase in the plastic strains is governed by:
\[ \dot{\varepsilon}_p = \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial \sigma} \] \hfill (2.9)

Where \( g_i \) is a function called plastic potential which depends of the type of flow rule chosen and \( \lambda_i \) is the plastic multiplier corresponding to \( g_i \).

### 2.1.3 Coupled plasticity–damage model

In the case where it is considered a coupled behaviour between the damage and the plasticity of the body, see for instance [33, 41], it has to be introduced in the damage model the following relationship:
\[ d = d(\varepsilon_p) \quad \text{and} \quad \bar{\sigma} = E_0 : \varepsilon_e \] \hfill (2.10)

using also for the plasticity model that:
\[ F(\bar{\sigma}, \varepsilon_p) \leq 0 \quad \text{and} \quad \dot{\varepsilon}_p = \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial \bar{\sigma}} \] \hfill (2.11)

### 2.2 One dimensional application of the model

The application of the previously seen governing equations to the one dimensional case leads to some simplifications that are going to be presented in this section.
2.2 One dimensional application of the model

2.2.1 Isotropic damage model

The damage model can be seen in figure 2.2 where the unloading–reloading stiffness varies as the damage develops in the material but no plastic strain appear. This behaviour is typical of some pseudoelastic materials like for instance the arterys [50].

\[ E = E_0 (1 - d) \]

Figure 2.2: Damage model

The equations that describe this behaviour are merely:

\[ \sigma = E \cdot \varepsilon \tag{2.12} \]
\[ E = E_0 (1 - d) \tag{2.13} \]

where \( d \) is again the damage parameter of the body and being the nominal stresses:

\[ \tilde{\sigma} = \frac{1}{1 - d} \cdot \sigma \Rightarrow \tilde{\sigma} = E_0 \cdot \varepsilon \tag{2.14} \]

2.2.2 Plasticity model

Concerning the plasticity model (see figure 2.3), the equations may be rewritten as follows:

\[ \varepsilon = \varepsilon_e + \varepsilon_p \tag{2.15} \]

\[ F(\sigma, \varepsilon_p) \leq 0 \Rightarrow \sigma = \sigma(\varepsilon_p) \tag{2.16} \]

This behaviour can for instance be found in the steel, where the plastic strains are not accompanied of a degradation in the stiffness modulus.

2.2.3 Coupled plasticity–damage model

Finally, and as presented in figure 2.4, both behaviours may be coupled. As a consequence, it is assumed that there is a relationship between the plastic strains developed by a specimen and its observed damage.

In this case, the governing equation may be written as follows:
Isotropic continuum models

This behaviour is typical of some materials like for instance the concrete, where the development of the plastic strains is due to the microcracking of the cement paste which also leads to a degradation in the stiffness modulus of the material.

In these materials, it is usual to divide the damage in a component due to the compressive plastic strains and other due to the tensile plastic strains. Furthermore, it is also possible to include the crack closure phenomenon, reducing the total damage parameter when a compression stress is applied to a specimen that has been previously cracked under tensile forces \[41\] as follows:

\[
d = d(\varepsilon_p) = d(\varepsilon_{p,c}, \varepsilon_{p,t}) = 1 - (1 - d_c(\varepsilon_{p,c}))(1 - p(\tilde{\sigma})d_t(\varepsilon_{p,t}))
\]  

(2.18)

where the damage parameter has been divided into two different variables, one controlling the damage under compression and the other the damage under tension. In the latter, it has been added an additional term which reduces the damage as tensile cracks close under
compressive stresses, whose its value can be obtained as follows$^1$:

\[ p(\tilde{\sigma}) = p_0 + (1 - p_0)r(\tilde{\sigma}) \]

where:
\[
\begin{align*}
  r(\tilde{\sigma}) &= 0 & \text{if } \tilde{\sigma} \leq 0 \\
  r(\tilde{\sigma}) &= 1 & \text{if } \tilde{\sigma} > 0
\end{align*}
\]

(2.19)

$^1$Good results have been obtained by Nechnech [41] with $p_0 = 0.1$ for uniaxial tests. However, this value has to take into consideration other parameters like for instance the presence of shear, long term or cyclic loading, ...
Chapter 3

Analytical model for the short–term behaviour of confined concrete

3.1 Introduction

This chapter presents an analytical model to reproduce the $\sigma - \varepsilon$ behaviour of concrete when a confinement pressure is applied to it, see figure 3.1.

Figure 3.1: Concrete specimen with lateral pressure

Certain analytical laws will be derived, based on the previously stated principles of continuum mechanics, in order to reproduce the response of concrete under monotonic loading as well as its unloading–reloading behaviour. The mechanical response of concrete will be studied for tension and compression taking into consideration the four possible loading cases (obtained combining $\sigma_{lat}$ and $\sigma_3$ with positive and negative values).

3.2 Tension behaviour

The first response that is going to be studied is the tensile behaviour of concrete. In order to do it, the response of a tie will be examined and later its behaviour will be generalised.
3.2.1 Tension tie

The study of the tensile response of concrete will be explained by means of a tension member using a discrete (cohesive) approach of the cracking phenomenon.

Figure 3.2 shows a general scheme of a tension tie where the cracking process is localized at a fracture process zone (FPZ) as proposed by Hillerborg [23].

Considering a perfect damage model, the global response of the tie can be reproduced using linear elastic springs whose behaviour is perfectly elastic up to its failure as can be seen in figure 3.3. This model assumes that, when the tie is loaded, the first element that fails is the weakest. At that moment, the total stress of the tie decreases although the resisting elements maintain their level of stress. Once the stress at the remaining elements is slightly increased, another one will fail and so on.

This model, however, can be improved if the springs are supposed not to behave linear elastically and they can develop plastic strains. With this type of constitutive law, the previously shown coupled plastic–damage response of figure 3.2 can be reproduced.

Moreover, the springs can also be supposed to present a rheological behaviour. If this is the case, and as it was previously seen, the stress over the different elements suffers a linear increase before the first crack is developed, after that, the stress suffers only a small increase in the remaining elements for the rest of the fracture process. This point has to be taken into consideration when the non–linear behaviour of the creep of concrete is going to be studied.

3.2.2 Continuum medium

The previous physical model for the tie can also be generalised for a continuum body by means of a lattice model (as proposed at the university of Delft [54]). With this type of approach, the cracking process can be studied for 2-D or 3-D problems as an extension of the tie model. Mainly, two different types of lattice models are proposed [9]: regular (cubic or hexahedric) meshes and irregular (random) meshes (see figure 3.4).

The cubic mesh represents an uncoupled behaviour in the different directions whereas the two others provoke a coupled response of the material (a certain damage in one direction affects
the behaviour in the other directions) depending mainly this effect of the lattice orientation. As a simplification, and taking into consideration the test results from Kupfer for biaxial loading [30], the tensile resistance in a given direction can be considered not to be affected by

Figure 3.3: Perfect damage model: (a) first cracking force; (b) stress state after cracking; (c) reloading process; (d) envelope behaviour
the stresses in the other. Then, an uncoupled behaviour will be assumed for concrete under tensile lateral stresses when a longitudinal tensile stress is applied, which can be interpreted as a cubic mesh for the lattice model.

3.2.3 Numerical model

Once the physical basis of the model has been proposed, in this section it is going to be explained the numerical model for the tension member.

Analytical description

The model assumes the presence of a discrete crack in a member of a given length $L$ and reproduces its monotonic and unloading–reloading behaviour. The model is also applicable when a number of cracks are distributed in a specimen with a given crack spacing. This idea, shown in figure 3.5, is exact when an equivalent length of the member is adopted as $L = s_c$.

![Figure 3.5: Cracking patterns for the model: (a) single crack in a short member; (b) multiple cracks at a distance $s_c$ in a long member.](image)

The response of the member can be divided into two parts. The first one corresponds to the elastic domain whereas the second one corresponds to the post–peak response.

If the stress–strain diagram of the member is studied$^1$, a linear behaviour is consider to be

$^1$In fact, this diagram may be used as a smeared crack approach, but it is preferred to be seen as the global response of a member with a discrete crack.
valid in the elastic domain even for unloading and reloading, thus:

\[ \sigma = E_c \varepsilon \longrightarrow \varepsilon = \frac{\sigma}{E_c} \quad (3.1) \]

Once the cracking process takes place, the concrete suffers an unloading process in the non-cracked zone and the crack opens as it was shown in figure 3.2. The complete response of the tie can be characterized if a softening law is defined. This law relates the crack opening displacement with the force that the crack is capable to transmit. In a general way, this law can be written as:

\[ \frac{\sigma}{f_{ct}} = \psi^{-1}(w) \longrightarrow w = \psi \left( \frac{\sigma}{f_{ct}} \right) \quad (3.2) \]

Then, for a given level of stress, it would be obtained that:

\[ \left\{ \begin{array}{l} \sigma_c = \sigma = E_c \varepsilon_c \\ \sigma_f = \sigma = f_{ct} \psi^{-1}(w) \end{array} \right. \quad (3.3) \]

Finally, summing up the increase in the length of the fracture process zone and the uncracked concrete, it is obtained that:

\[ \varepsilon = \frac{\Delta L_c + \Delta L_f}{L} = \frac{\sigma}{E_c} + \frac{\psi \left( \frac{\sigma}{f_{ct}} \right)}{L} \quad (3.4) \]

This equation governs the decay of the member in the stress–strain response. As can be seen in figure 3.6, depending on the length of the member and the softening law, the curve may suffer a snap–back phenomenon if a displacement control is chosen for elements whose length is quite big [37].

For the unloading and reloading of the concrete, it is adopted a coupled plasticity–damage model in the fracture process zone. In it, the increase of the plastic crack width opening is defined directly proportional to the decrease of the stiffness of the unloading–reloading curve, thus:

\[ w_p = \left( \frac{1}{K} \right) (\lambda f_{ct}) \quad (3.5) \]
where \( \lambda \) is a coefficient whose value is proposed to be adopted as \( \lambda \approx 0.60 \). Obviously, if \( \lambda = 0 \) then the model will reproduce a perfect damage behaviour.

Figure 3.7 (a) shows the relationship between the plastic displacement and the stiffness parameter as defined by equation (3.5). This relationship can also be seen as the rotation of the loading–reloading line over a fix point if the \( \sigma – w \) curve is plotted. This geometrical interpretation, shown in figure 3.7 (b), has been named by many authors as the focal point theory\(^2\).

![Diagram](image)

**Figure 3.7:** Coupled plasticity–damage model: (a) coupled parameters; (b) focal point (geometrical) approach

Finally, the plastic strain after the complete unload of an element can be obtained as follows:

\[
\frac{w}{w_p} = \frac{\sigma + \lambda f_{ct}}{\lambda f_{ct}}
\]

(3.6)

\[
w_p = w \frac{\lambda f_{ct}}{\sigma + \lambda f_{ct}}
\]

(3.7)

In light of this result, it is finally obtained that:

\[
\varepsilon_p = \frac{w_p}{L} = \psi \left( \frac{\sigma}{f_{ct}} \right) \frac{\lambda}{\frac{\sigma}{f_{ct}} + \lambda}
\]

(3.8)

A general scheme of the model with all the characteristic points for the loading and unloading–reloading behaviour can be seen in figure 3.8.

Once the \( \sigma – \varepsilon \) curve of the tie is obtained, other results like for instance the \( F – \delta \) curve can be directly derived using the geometrical properties of the member (area and length). Also, the crack width opening \( (w) \) is perfectly known at each point.

**Application of a softening law and comparison with test results**

In order to show an application of the model, a softening law will be implemented in order to define the \( \psi \) function. Many laws have been proposed for this curve (linear, bilinear, logarithmic, . . .). In this study, for the monotonic loading, it is going to be used the law proposed by Hordijk due to its good agreement with the experimental data [31].

\(^2\)For instance, Yankelevsky and Reinhardt [59], propose a value for the third focal point \( (z_3) \) equal to \( 0.75f_t \) which in this case will be the value that corresponds to \( \lambda \).
3.2 Tension behaviour

\[ \epsilon = \frac{\sigma}{E_c} = \psi \left( \frac{\sigma}{E_c} \right) \]

\[ \epsilon_p = \psi \left( \frac{\sigma}{E_c} \right) + \frac{\lambda}{E_c \lambda + \lambda} \]

Figure 3.8: Tension tie response

According to Hordijk:

\[ \frac{\sigma}{f_c} = \left( 1 + \left( c_1 \frac{w}{w_c} \right)^3 \right) \exp \left[ -c_2 \frac{w}{w_c} \right] - \frac{w}{w_c} (1 + c_1^3) \exp[-c_2] \tag{3.9} \]

where:

\[
\begin{align*}
  c_1 &= 3.0 \\
  c_2 &= 6.93 \\
  w_c &= 5.14 \frac{G_F}{f_{ct}} 
\end{align*}
\tag{3.10}
\tag{3.11}

for the value of \( G_F \), it is adopted the values proposed in the MC-90:

\[ G_F = G_{F0} \left( \frac{f_{cm}}{f_{cm0}} \right)^{0.7} \rightarrow \begin{cases} f_{cm0} = 10 \text{ MPa} \\ f_{cm} \leq 80 \text{ MPa} \end{cases} \tag{3.12} \]

being the values of \( G_{F0} \) the ones shown in table 3.1.

<table>
<thead>
<tr>
<th>( d_{max} ) [mm]</th>
<th>8.0</th>
<th>16.0</th>
<th>32.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_{F0} ) [N/mm]</td>
<td>0.02</td>
<td>0.03</td>
<td>0.058</td>
</tr>
</tbody>
</table>

Table 3.1: \( G_{F0} \) values

The results obtained with the analytical model\(^3\) compared to those obtained by Reinhardt & Cornelessen [11] for a tensile tie are shown in figure 3.9 where for the softening law it has been adopted \( G_{F0} = 0.03 \).

Further comparison of the model has also been performed with the experimental results of Gopalaratnam & Shah [21] taken from [41]. The comparison between the experimental curve and the analytical model can be seen in figure 3.10. Again, good agreement is found between the analytical model and the experimental results\(^4\).

\(^3\)The response curve has been obtained adopting \( w \) as a dummy variable. With it, the value of the coefficient \( K \) is obtained and finally, using the value of \( K \), it is obtained the value of \( w_p \).

\(^4\)In this case the test was stopped when the tie was only at a damage parameter of \( d = 0.73 \).
Figure 3.9: Comparison of the analytical model with the test results from Reinhardt & Corneillessen [11]: (a) experimental results; (b) loading–reloading cycles; (c) stiffness modulus as a function of the plastic displacement; (d) damage parameter as a function of the plastic displacement.

Figure 3.10: Comparison of the analytical model with the test results from Gopalaratnam & Shah [21] taken from [41]: (a) loading–reloading cycles; (b) stiffness modulus as a function of the plastic displacement; (c) damage parameter as a function of the plastic displacement.
3.3 Compression behaviour

Finally, the model has also been applied to the experimental results obtained by Denarié [14] over tensile wedge specimens. A scheme of the tested specimens can be seen in figure 3.11 where the points named A and B were the points where the load was applied by means of a wedge.

![Figure 3.11: Wedge tensile specimen from Denarié [14]](image)

In this case, the model is not exact because the test specimen is not a tension tie. However, fixing a value for the tensile resistance of the concrete, some equivalent length and area properties can be defined for the $F - w$ curve at the fracture process zone in order to obtain an equivalent stress–strain curve\(^5\). With this methodology, the damage and stiffness modulus relationships can be obtained as can be seen in figure 3.12.

### 3.3 Compression behaviour

#### 3.3.1 Monotonic loading

Concerning the compressive behaviour of concrete, the triaxial state of stresses as well as the transverse strains play a critical role in the response of the material. Without lateral confinement, and due to the different stiffness between the cement matrix and the aggregates, the load is transmitted mainly due to the development of a strut and tie resistant mechanism (figure 3.13). This explains not only the lateral expansion but also the longitudinal cracking that takes place in the element.

Obviously, when a lateral confining pressure is applied to the specimen, the tie behaves as if it were prestressed, and so, it is capable of bearing higher longitudinal stresses before the cracking process begins. Also, the failure mode changes, and instead of a smeared longitudinal cracking, an inclined crack develops with the sliding of two different bodies (see for instance [55]). To the limit, the concrete can be subjected to a uniform tricompression state where no direction is in principle privileged in the cracking process.

The change in the failure mode does not only affect in the compressive resistance but also in the ductility of the specimen. As can be seen in figure 3.15, a concrete without confinement pressure presents a certain brittleness (increasing for higher resistances), this type of

\(^5\)The value of the equivalent parameters (area and length) should change with the level of cracking in the specimen. However, for the different analyses performed, they have been kept constant, obtaining a reasonably good response. For the tensile resistance of the concrete, it has been adopted a mean value of $f_{ct} = 3$ MPa has been adopted.
Figure 3.12: Comparison of the analytical model with the test results from Denarié [14]: (a) experimental results; (b) loading-reloading cycles; (c) stiffness modulus as a function of the plastic displacement; (d) damage parameter as a function of the plastic displacement.

Figure 3.13: Development of longitudinal cracking without confinement pressure, adapted from Muttoni [39]: (a) strut and tie mechanism; (b) cracking pattern for $0.8f_c \leq \sigma \leq f_c$; (c) cracking pattern for $\sigma \geq f_c$.

Response can be identified as typical of cohesive materials. On the other hand, when a lateral confinement pressure is applied, ductility increases remarkably, being this response typical of frictional materials.
3.3 Compression behaviour

Figure 3.14: Compressive failure patterns: (a) without lateral confinement, \( h \sim 2\phi \); (b) with lateral confinement, \( h \sim \phi \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \), where \( \phi \) is the frictional angle of the concrete

Figure 3.15: Compressive failure response for confined and unconfined concrete

Based on these physical mechanisms, an analytical law can be proposed in order to simulate the monotonic loading response of confined concrete.

Considering the case where no transverse pressure is applied to the concrete specimen, the Thorenfeldt, Tomaszewicz and Jensen law \([57]\) provides good agreement with test results \([10]\), being its general expression as follows:

\[
\sigma_3 = f_c \left[ \frac{n \left( \frac{\varepsilon_3}{\varepsilon_c} \right)}{n - 1 + \left( \frac{\varepsilon_3}{\varepsilon_c} \right)^n k} \right] \tag{3.13}
\]
where, according to the authors, \( n = 0.8 + \frac{f_c (\text{MPa})}{f_t} \); being also \( \varepsilon'_c = \frac{f_c}{E_c} \cdot \frac{n}{n-1} \) and \( k = 0.67 + \frac{f_c (\text{MPa})}{62} \) \( \geq 1.0 \). Using the definition of \( \varepsilon'_c \), equation 3.13 can be rewritten in the following way:

\[
\sigma_3 = \frac{\varepsilon_3 E_c}{1 + \left( \frac{\varepsilon_3}{(n-1)(\varepsilon'_c)^n} \right)} \quad (3.14)
\]

However, this law does not provide good results when it is applied to study the response of confined specimens. In this case, the compressive strength increases following, according to Richart\(^6\) [49], a relationship close to:

\[
f_{cc} = f_c + k \sigma_{lat} \rightarrow k \approx 4.0
\]

(3.15)

Accepting that for a tricompession state of stresses the element can overcome big plastic strains, and assuming that the elastic modulus is given by the Poisson’s relationship \( E_{cc} = \frac{E_c}{1-2\nu} \) it is proposed for this state to use a modified Ramberg–Osgood law with constant yield stress as follows:

\[
\sigma_3 = \frac{\varepsilon_3 E_{cc}}{1 + (B\varepsilon_3)} \quad (3.16)
\]

where for \( B \) it has to be adopted\(^7\) \( B = -\left( \frac{E_{cc}}{f_{cc}} \right) \); being \( f_{cc} = 5f_c \) using (3.15).

It can be seen that the previous equations describing the unconfined (3.14) and confined behaviour (3.16) are very similar. Due to this reason, it is proposed a generalized law that coincides with them in the limit cases and represents a smooth transition in the rest. The law can be written as:

\[
\sigma_3 = \frac{\varepsilon_3 E_{cc}}{1 + \left( \frac{5 \varepsilon_3}{50} \right)} \quad (3.17)
\]

Where the following values are adopted:

\[
\varepsilon_0 = \frac{n f_{cc}^*}{E_{cc} \cdot (n-1)(\frac{\alpha + 1}{n})}
\]

\[
\gamma = \frac{\sigma_{lat}}{f_c}
\]

\[
\xi = \gamma \text{ if } \gamma \leq 1.0 \quad (\xi = 1.0 \text{ rest})
\]

\[
E_{cc} = \frac{E_c}{1-2\nu\delta} \text{ where } \delta = \frac{\sigma_{lat,max}}{f_c} \text{ in a proportional loading case}^{8}
\]

\[
\beta = \frac{8.5-7.0\xi}{f_c}
\]

\[
\alpha = nk + (1-nk)\gamma^\beta
\]

\[
f_{cc}^* = f_c \left[ 1 + \gamma \left( \frac{5}{n^2} - 1 \right) \right]
\]

\(^6\)Which in fact is a Mohr–Coulomb failure criterion. Although other failure criteria (Newman, Leon, ...) are more accurate, the results obtained applying a Mohr-Coulomb criterion are satisfactory enough, see for instance [55].

\(^7\)Taking \( \varepsilon \) negative for compression values.

\(^8\)Where the lateral pressure is increased until the compressive strength is reached. Different values for \( \delta \) are possible for other loading sequences.
where \( n \) and \( k \) are the TTJ coefficients whose values, slightly different from those proposed by the authors, are adopted as follows:

\[
\begin{align*}
    n &= 0.8 + \frac{f_c [\text{MPa}]}{22} \\
    k &= 0.6 + \frac{f_c [\text{MPa}]}{50}
\end{align*}
\]

The results obtained with this expression can be seen in figure 3.16 for the test results obtained by Kosaka et al. [29] and taken from Muttoni [39]. It should be noticed that the values obtained for the compressive resistance \( f_{cc} \) with the formula of Richart do not agree well in this case with the measured values as can be seen in table 3.2. Due to this reason, an equivalent lateral stress\(^9\) has been applied in the analytical equation in order to have the same compressive resistance as the one measured for the concrete according to equation (3.15). Taking into consideration this point\(^{10}\), good agreement is found between the analytical expression and the experimental measures, even for high level of strains \( \varepsilon_3 \).

<table>
<thead>
<tr>
<th>( \sigma_{lat} ) applied Kosaka [MPa]</th>
<th>0.0</th>
<th>-0.1</th>
<th>-0.3</th>
<th>-0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{cc} ) measured Kosaka [MPa]</td>
<td>27.0</td>
<td>29.0</td>
<td>32.0</td>
<td>34</td>
</tr>
<tr>
<td>( f_{cc} ) according to (3.15)</td>
<td>27.0</td>
<td>27.4</td>
<td>28.2</td>
<td>29.4</td>
</tr>
</tbody>
</table>

Table 3.2: Values of \( f_{cc} \)

![Figure 3.16: Compression behaviour for a concrete of \( f_c = 27 \) MPa, test results from Kosaka et al. [29], taken from Muttoni [39]: (a) test results; (b) comparison with analytical model](image)

Figure 3.17 shows the comparison of the analytical model with the test results from Imran and Pantazopoulou [25]. The model gives again a reasonable estimation of the whole response of the concrete, even for high levels of confining pressure and axial strains.

Also, the experimental results obtained by Richart [49] for a concrete of \( f_c = 17.7 \) MPa and \( f_c = 25.2 \) MPa are presented in figures 3.18 and 3.19 respectively. Again, good agreement is found even for high levels of lateral confinement.

Finally, the model has also been verified for high performance concrete using the tests results from Candappa et al. [7] for compressive strengths ranging from 40 to 100 MPa. The results

\(^9\)Of approximately double value of the one measured by Kosaka.

\(^{10}\)This behaviour is also in accordance with the experimental measures of Candappa et al. [7] where for low levels of confinement a value for \( k \) of 5.3 (higher than 4.0) is proposed.
Figure 3.17: Compression behaviour, test results from Imran and Pantazopoulou [25]: (a) $f_c = 27$ MPa and $\sigma_{lat} = 0$ MPa; (b) $f_c = 45$ MPa and $\sigma_{lat} = -8.6$ MPa; (c) $f_c = 60$ MPa and $\sigma_{lat} = -38.4$ MPa

Figure 3.18: Compression behaviour, test results from Richart [49] for a concrete of $f_c = 17$ MPa, taken from Muttoni [39]: (a) test results; (b) comparison with analytical model

Figure 3.19: Compression behaviour, test results from Richart [49] for a concrete of $f_c = 25$ MPa, taken from Sigrist [56]: (a) test results; (b) comparison with analytical model
are shown in figures 3.20 obtaining again good agreement between the theoretical predictions and the test results\textsuperscript{11}.

![Figure 3.20: Comparison of the analytical model with the test results of Candappa et al.\textsuperscript{7} for high performance concrete. The different curves in each graphic correspond to confinement pressures of 0, 4, 8 and 12 MPa: (a) $f_c = 40$ MPa; (b) $f_c = 75$ MPa; (c) $f_c = 100$ MPa](image)

### 3.3.2 Lateral expansion

Once the behaviour in the axial direction has been studied, it has to be analyzed the transverse response of a specimen in order to complete the monotonic loading model of confined concrete. Some experimental research has been performed on this field showing certain evidences:

- The concrete exhibits first an elastic behaviour up to $\sigma_3 \leq (0.7 - 0.8) f_c$ with a Poisson’s coefficient, almost constant at this stage, around $\nu_c = 0.2$
- Once the concrete starts to microcrack, the Poisson’s coefficient increases, reaching a value of $\nu_c = 0.5$ (no volumetric strain) for $\sigma_3 = f_{cc}$
- After the peak has been reached, the Poisson’s coefficient continues to increase

Different theoretical approaches have been proposed in order to model the lateral expansion of concrete. For instance, Pantazopoulou et al.\textsuperscript{43, 25} propose to relate the volumetric strain with the axial strain using a cubic function ($\varepsilon_v = \varepsilon_v(\varepsilon_3)$). However, other researchers like Candappa et al.\textsuperscript{7} express this relationship using the Poisson’s coefficient and the stress in the specimen with two polynomial expressions\textsuperscript{12} ($\nu = \nu(\sigma_3)$).

In this work, it is proposed to consider a linear relationship between the Poisson’s coefficient and the axial strains of the specimen ($\nu = \nu(\varepsilon_3)$) once the microcracking process starts, see figure 3.21.

The law can be written analytically in the following way:

$$
\begin{align*}
\varepsilon_3 &\leq \varepsilon_{3,c} \quad \nu = \nu_e \\
\varepsilon_3 &> \varepsilon_{3,c} \quad \nu = \nu_e + (\varepsilon_3 - \varepsilon_{3,c}) \frac{(\nu_p - \nu_e)}{\varepsilon_{3,p} - \varepsilon_{3,c}}
\end{align*}
$$

\textsuperscript{11}As it was previously stated in the tests of Kosaka, the tests performed by Candappa do not show good agreement with the prediction of the ultimate strength of Richart. Then, an equivalent lateral pressure of approximately 1.25 times the one applied by the authors has been used in the model in order to obtain the same compressive resistance.

\textsuperscript{12}One for the pre-peak behaviour and other for the post-peak behaviour.
where:

- $\nu_e$ is the value of the elastic Poisson’s coefficient ($\nu_e \approx 0.2$)
- $\nu_p$ is the value of the Poisson’s coefficient when the maximum stress is reached ($\nu_p \approx 0.5$)
- $\varepsilon_{3,c}$ is the value of the strain in the concrete when the microcracking process starts ($\varepsilon_{3,c} \approx 0.8 \frac{f_c}{E_c}$)
- $\varepsilon_{3,p}$ is the value of the strain in the concrete when the maximum stress is reached.

Deriving in equation \((3.17)\), it is obtained that the maximum stress is obtained for a strain equal to $\varepsilon_{3,p} = \frac{\varepsilon_0}{(\alpha - 1)\pi}$.

The volumetric strain can be calculated if it is taken into consideration that
\[\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = (1 - 2\nu) \left( \frac{\sigma_3}{E} + 2 \frac{\sigma_{lat}}{E} \right)\]  
(3.19)

and defining an apparent axial Poisson’s coefficient for confined cases as follows:
\[\varepsilon_v = \frac{\sigma_3}{E}(1 - 2\nu^*) = \varepsilon_3(1 - 2\nu^*)\]  
(3.20)

it is obtained from equations \((3.19)\) and \((3.20)\) that:
\[\nu^* = \nu - \frac{\sigma_{lat}}{\sigma_3}(1 - 2\nu)\]  
(3.21)

The results for the volumetric strain obtained with this law are compared with the experimental tests performed by Imran and Pantazopoulou \cite{25} in figure 3.22 showing good agreement\(^{14}\). The curves for confinement pressures of 0 and 38.4 MPa are reproduced better by the model than the one corresponding 8.6 MPa. The reason for this is found in the position of the axial strain at peak load, slightly lower in the analytical model than what was found in the experimental test for that specimen.

\^13Where the stiffness modulus is considered to be equal in all directions.

\^14It has to be noticed that the test results show a volumetric strain different to zero before the axial strain is applied. This fact is due to a non-proportional loading process of the specimens, applying first the confinement pressure and later the axial stress. In contrast to this situation, the analytical prediction has been obtained applying proportional loading and so the volumetric strain is zero when the axial strain is also zero. In any case the differences due to this effect are small.
3.3 Compression behaviour

Figure 3.22: Comparison of the analytical model with the test results from Imran and Pantazopoulou [25]: (a) $f_c = 27$ MPa and $\sigma_{lat} = 0$ MPa; (b) $f_c = 45$ MPa and $\sigma_{lat} = -8.6$ MPa; (c) $f_c = 60$ MPa and $\sigma_{lat} = -38.4$ MPa

Also, figure 3.23 shows the results of the Poisson’s coefficient for the same tests of Imran and Pantazopoulou [25] expressed in terms of the axial strain. In these diagrams it can be noticed that effectively the Poisson’s coefficient follows almost a straight line once the cracking process starts, showing good correlation with the theoretical model.

Figure 3.23: $\nu$ coefficient from the test results of Imran and Pantazopoulou [25] expressed as a function of the axial strain: (a) $f_c = 27$ MPa and $\sigma_{lat} = 0$ MPa; (b) $f_c = 45$ MPa and $\sigma_{lat} = -8.6$ MPa; (c) $f_c = 60$ MPa and $\sigma_{lat} = -38.4$ MPa

Finally, the model has been compared with the test results from Candappa et al. [7], see figure 3.24, where the Poisson’s coefficient is plotted against a coefficient called $\beta = \frac{\sigma_3}{f_{cc}}$ by the authors. Good agreement is again found between the analytical prediction and the test results.

Some interesting conclusions can be directly derived from this model. For instance, the higher the strength of the concrete, the higher the volumetric expansion is experienced by the concrete. On the other hand, the lower the strength of the concrete (or the higher the confinement pressure), the lower the volumetric expansion is suffered.

In reference [7], the lateral pressure of each experimental curve $\beta - \nu$ is not specified. However, they have to be between 4 and 12 MPa because no post–peak measures were performed for unconfined specimens.
Figure 3.24: Comparison of the analytical model (continuous lines) with the test results from Candappa et al. [7] (dashed lines): (a) $f_c = 40$ MPa and $\sigma_{lat}$ ranging from 4 to 12 MPa; (b) $f_c = 100$ MPa and $\sigma_{lat}$ ranging from 4 to 12 MPa

3.3.3 Unloading–reloading behaviour

Concerning the unloading and reloading behaviour of the concrete, a coupled damage–plasticity model will also be used.

Many differences have to be taken into consideration from the tension model previously seen while studying the response of the concrete under compression. This time, the failure zone is not located at a certain position and can be considered to be distributed over a certain length (see figure 3.14) where some parts of the damaged zone will remain under elastic conditions whereas others will overcome a high degree of damage and plastic strains. The behaviour can then be divided again into two parts, one showing a perfect elastic behaviour and other showing a degradation of its properties with the development of anelastic strains. This scheme, shown in figure 3.25, can be compared to the one previously developed for traction but in terms of the stress-strain diagram in order to represent the smeared nature of the problem.

According to Rodríguez [50], under pseudoelastic conditions, the increase in the damage parameter is only due to the deviatoric part of the density energy function:

\[ W = W_{\text{vol}} + (1 - d)W_{0,\text{iso}} \]  

Thus, it can be assumed that in principle a negligible damage will occur under a hydrostatic state of stresses. This consideration, although not exact for concrete, shows good agreement with experimental results. For instance, after a microscopic study of the surface of concrete subjected to high confinement pressures, Sfer et al. [55] found that the closure of voids and microcracks were probably the reason of a very slow decrease (almost negligible) of the unloading–reloading stiffness. As a consequence, it will be assumed that the damage experienced when a hydrostatic state of stresses is applied to the concrete can in principle be neglected in comparison to that of a deviatoric state.

If the case of a concrete specimen subjected to an uniaxial state of stresses is studied, then it is found that there is a certain deviatoric component in the stress tensor:
as a consequence, a certain increase of the damage will be developed. On the other hand, if an hydrostatic state of stresses is applied to the concrete, there will be no deviatoric term of the stress tensor.

Based on these points, the study of the unloading–reloading behaviour will be divided into two parts, one comprising the behaviour without lateral pressures and other considering them. The aim of this approach is to study first an adequate law for modeling the damage in the concrete when a certain deviatoric component is considered, and then to study a transition to the ideal state where the damage can be assumed to be negligible.

**Unconfined response**

Considering the longitudinal stresses and strains of the concrete, and naming as $K$ the damage stiffness in analogy to the tension case, it is obtained that:

$$
\varepsilon = \varepsilon_e + \varepsilon_p \tag{3.24}
$$

Where, according to figure 3.25, it can be obtained that:

$$
\varepsilon = \varepsilon_{el0} + \Delta \varepsilon + \varepsilon_p \tag{3.25}
$$

and taking into consideration that:

$$
\begin{cases}
\sigma = E(\varepsilon_{el0} + \Delta \varepsilon) \\
\sigma = E_0 \varepsilon_{el0}
\end{cases} \tag{3.26}
$$
It is obtained that:

\[
\frac{E}{E_0} = \frac{\varepsilon_{el0}}{(\varepsilon_{el0} + \Delta \varepsilon)} \quad (3.27)
\]

and so, the damage parameter has a value of:

\[
d = 1 - \frac{\varepsilon_{el0}}{\varepsilon_{el0} + \Delta \varepsilon} = 1 - \frac{1}{1 + \frac{\Delta \varepsilon}{\varepsilon_{el0}}} \quad (3.28)
\]

At this point, and similarly as it was done in the tensile case, a hypothesis will be done on the relationship between the different strains. The adopted one is the following:

\[
\frac{\varepsilon_{an}}{\sigma + \lambda f_c} = \frac{\varepsilon_p}{\lambda f_c} \quad (3.29)
\]

this relationship can also be interpreted in a geometrical way\(^{16}\) as it is shown in figure 3.26. Using it, and taking into consideration that \(\varepsilon_{an} = \varepsilon_p + \Delta \varepsilon\) (figure 3.25) it is obtained that:

\[
\Delta \varepsilon = \left(\frac{\sigma + \lambda f_c}{\lambda f_c} - 1\right) \varepsilon_p = \frac{\sigma}{\lambda f_c} \varepsilon_p \quad (3.30)
\]

Finally, entering with \(\sigma = E_0 \varepsilon_{el0}\) and equation (3.30) in equation (3.28) it is obtained that:

\[
d = 1 - \frac{1}{1 + \varepsilon_p \frac{E_0}{\lambda f_c}} \quad (3.31)
\]

where a reasonable value for \(\lambda\) has been found after a parametric study around \(\lambda = 4\).

As a calculation procedure, it is proposed to adopt \(\varepsilon\) as a dummy variable, then using the constitutive law for the concrete, equation (3.13) in this case, it is obtained the value of \(\sigma\). Later, it is computed \(\varepsilon_{an} = \varepsilon - \frac{\varepsilon_{an}}{\lambda f_c}\), where \(\varepsilon_{an}\) is the anelastic strain of the concrete. With this value, it is obtained the plastic strain as \(\varepsilon_p = \varepsilon_{an} \frac{\lambda f_c}{\lambda f_c + \sigma}\), allowing now to obtain the rest of the parameters of the model.

The results obtained applying this relationship are shown in figure 3.27 for the test results from Karsan [38] (taken from Nechnech [41]) comprising several unloading–reloading cycles in a concrete without lateral stresses.

\(^{16}\)In this case, it corresponds to a focal point but this time in the \(\sigma - \varepsilon_{an}\) curve.
3.3 Compression behaviour

Figure 3.27: Comparison of the analytical model with the test results from Karsan [38] (taken from Nechnech [41]) without lateral pressure: (a) loading–reloading cycles; (b) damage parameter as a function of the plastic displacement.

Also, figure 3.28 shows the results obtained from Imran and Pantazopoulou [25] for an specimen without confining pressure. Good agreement is found in both cases between the theoretical model and the experimental results.

Confined response

When dealing with a concrete confined by lateral pressures, the failure mode changes due to the increase of the volumetric stress and the reduction in the deviatoric part of the stress tensor.

Assuming that under confinement pressure the damage experienced by the concrete is lower than in an unconfined situation, it is proposed to adopt the following relationship:

\[
d = 1 - \frac{1}{1 + \varepsilon_p \frac{f_0}{\lambda c}}
\]  

(3.32)

being \( \lambda_{cc} = \frac{\lambda}{(1 - \xi)^2} \); where \( \lambda \) was studied in the unconfined case where it was proposed a value of \( \lambda \approx 4 \) and \( \xi \) was defined for equation 3.17. Also, the calculation procedure for obtaining the stress–strain response of the concrete comprising the unloading–reloading cycles is the same as in the unconfined case.
In equation (3.32) it has to be noticed that no damage is developed when it is satisfied that $\sigma_{\text{lat}} \geq f_c$. This condition has to be understood in the sense that the damage is negligible for high levels of confinement, although it would be theoretically zero only for $\sigma_{\text{lat}} = f_{cc}$.

Some results obtained using this relationships can be seen in figures 3.29 and 3.30 where a reasonable agreement is found for both cases (ranging $\frac{\sigma_{\text{lat}}}{f_c} \in (0.2; 0.6)$).

Figure 3.29: Comparison of the analytical model with the test results from Imran and Pantazopoulou [25] when a lateral pressure of $-8.6$ MPa is applied ($\frac{\sigma_{\text{lat}}}{f_c} \approx 0.2$): (a) loading–reloading cycles; (b) damage parameter as a function of the plastic displacement.

Figure 3.30: Comparison of the analytical model with the test results from Imran and Pantazopoulou [25] when a lateral pressure of $-38.4$ MPa is applied ($\frac{\sigma_{\text{lat}}}{f_c} \approx 0.6$): (a) loading–reloading cycles; (b) damage parameter as a function of the plastic displacement.

Beyond the numerical results, it is interesting to notice the qualitative performance of the proposed model: the higher the lateral pressures, the slowest the damage parameter increases\(^{17}\). The contrast of this hypothesis with the experimental results from Imran and Pantazopoulou seems to provide good agreement.

**Lateral expansion**

Also, the lateral expansion shows an unloading–reloading behaviour. The experimental tests performed by Imran and Pantazopoulou on this topic, see the volumetric strain diagrams in figure 3.31, show two different stages:

\(^{17}\)To the limit, if an hydrostatic state of stresses were applied to the concrete specimen, no damage will take place.
3.3 Compression behaviour

Figure 3.31: Test results from Imran and Pantazopoulou [25] with loading–reloading cycles and confined concrete

1. A first domain, for strains lower than $\varepsilon_3 \leq \varepsilon_{3,c}$, with an elastic behaviour (the unloading–reloading loops are parallel to the monotonic loading path). This condition can be expressed as:

$$\varepsilon_3 \leq \varepsilon_{3,c} \rightarrow \nu^* = \nu^*_e$$

(3.33)

2. A second domain, once the cracking process has started, where the volumetric strain is almost constant (with a loop parallel to the horizontal axis) that can be expressed as:

$$(1 - 2\nu^*_l)\varepsilon_{3,l} = (1 - 2\nu^*_u)\varepsilon_{3,u} \rightarrow \nu^*_u = \frac{1}{2} \left( 1 - \frac{\varepsilon_{3,l}}{\varepsilon_{3,u}} (1 - 2\nu^*_l) \right)$$

(3.34)

where $\nu^*_l$ is the apparent Poisson’s coefficient in the monotonic loading branch as defined in equation (3.21) and $\nu^*_u$ is the apparent Poisson’s coefficient once the unloading process has finished. Also, $\varepsilon_{3,l}$ is the strain before the unloading process starts and $\varepsilon_{3,u}$ is the strain after the unloading process has taken place.

The different values proposed for the apparent Poisson’s coefficient ($\nu^*$) in the two domains can be seen in figure 3.32.

3.3.4 Application of the model to fibre reinforced concrete

Although the model has been developed for normal strength concrete, it can also be applied to high strength concrete. In order to confirm this point, the failure pattern of the latter is studied.

Figure 3.33 shows the failure in a specimen of $f_c = 170$ MPa of compressive resistance, where picture (a) presents the failure mode of the concrete without fibers and picture (b) the pattern when the concrete is reinforced with a certain amount of fibers [27].

It is interesting to notice the analogy between the failure patterns shown in figure 3.14 for normal strength concrete when lateral confinement was present or not. It can then be identified that the high strength concrete fails as a normal concrete without lateral pressure when
it does not have fibres, whereas the failure mode when the concrete has fibres is similar to that of normal concrete with lateral confinement.

Taking this point into consideration, the high strength concrete without fibres could in principle be modeled assuming in equation (3.14) that $\sigma_{lat} = 0$. Doing so, the result obtained is shown in figure 3.34.

A pretty good estimation of the response of the curve is obtained, specially in order to reproduce the brittle failure of the concrete. Near the peak, the model is stiffer than the experimental results, but in any case the analytical prediction can be considered valid.

The response of the concrete when fibers are added to the cement matrix can also be reproduced with the model if the presence of a lateral confinement pressure is considered. This lateral pressure is due to the fibres, whose effect is active once the concrete cracks. The process starts with the development of the microcracking around $(0.8 - 0.9) f_c$. At this loading rate, several cracks develop and the fibers between the lips of them become active. The activation of these fibers induces a lateral confinement state that increases as the cracking process develops and the diagonal macrocrack forms. An analytical law has been developed in order to reproduce an equivalent confinement pressure as follows:
3.3 Compression behaviour

Figure 3.34: Comparison of the analytical model with the test results of Jungwirth and Muttoni [27] for ultra–high performance concrete without fibers: (a) test results; (b) comparison with analytical model with $\sigma_{lat} = 0$.

$$
\sigma_{lat} = \begin{cases} 
\sigma_{pre} \left( \frac{\varepsilon_3}{\varepsilon_p} \right)^3 & \text{if } \varepsilon_3 \leq \varepsilon_p \\
\sigma_{pre} + \sigma_{pos} \left( 1 - \exp \left[ -\frac{\varepsilon_3 - \varepsilon_p}{3\varepsilon_3} \right] \right) & \text{if } \varepsilon_3 > \varepsilon_p 
\end{cases}
$$

(3.35)

Figure 3.35 shows the results obtained applying the previous ideas and adopting $\varepsilon_p = 0.005$; $\sigma_{pre} = 4.0$ and $\sigma_{pos} = 18.0$ to obtain the response of a fibre reinforced concrete of ultra–high performance. The experimental tests where carried out by Jungwirth and Muttoni [27] over the same specimens previously studied in figure 3.34.

As can be seen from that figure, the post–peak behaviour changes dramatically. Mainly, the ductility of the concrete increases in such a way that it can even handle strains greater than $-1.6\%$, whereas without fibres, the maximum strain reached by the concrete was about $-0.55\%$.

Also, figure 3.36 shows the results obtained by Jungwirth and Muttoni in [26] for a concrete of 130 MPa, adopting for the model the following values: $\varepsilon_p = 0.005$; $\sigma_{pre} = 1.0$ and $\sigma_{pos} = 7.0$. The model is accepted in order to reproduce the behaviour of ultra–high performance concrete with fibres and to give an explanation of the observed failure modes. However, it has not been fully verified and further study of the problem is required.

### 3.3.5 Application of the model to steel encased concrete

In order to study the applicability of the model to other cases, it is proposed in this section to verify its behaviour for modeling steel encased concrete columns. Compared with unconfined specimens, this type of column exhibits a more ductile behaviour and higher ultimate loads. These two effects can be explained taking into consideration the phenomenon of the lateral confinement produced by the steel tube over the encased concrete.

Figure 3.37 (adapted from [32]) shows the results obtained for three different types of concrete when it is encased in tubes of increasing thickness. From that figure, it can clearly be seen that the thicker the tube, the higher the external load that the column can bear.

Also it is noticeable the effect in the shape of the response curve that the compressive resistance of the concrete has. For normal concrete, a very smooth decay branch after that maximum load is observed or even no decay at all. On the other hand, for high or ultra–high
Figure 3.35: Comparison of the analytical model with the test results of Jungwirth and Muttoni [27] for ultra–high performance concrete with fibers ($\rho_f = 2.5\%$): (a) contributing fibers to the internal transverse pressure; (b) development of transverse pressure with increasing strains; (c) test results; (d) comparison with analytical model

Figure 3.36: Comparison of the analytical model with the test results of Jungwirth and Muttoni [26] for ultra–high performance concrete with fibers ($\rho_f = 2.0\%$): (a) test results; (b) comparison with analytical model; (c) development of transverse pressure with increasing strains

performance concrete, a strong decay is observed just after the maximum load, followed by a branch of constant load\(^\text{18}\).

The problem when studying the response of steel confined sections is the determination of

\(^\text{18}\)This results has also been confirmed by other researchers for normal and ultra–high performance concrete, see for instance [58].
3.3 Compression behaviour

Figure 3.37: Experimental results by Lahlou and Lachemi [32] for normal, high and ultra-high performance concrete encased in a steel tube: (a) general scheme of the specimens; (b) $N - \delta$ response for $f_c = 50$ MPa; (c) $N - \delta$ response for $f_c = 90$ MPa; (d) $N - \delta$ response for $f_c = 130$ MPa

the lateral stresses. For instance, according to the authors of the previously seen results [32], not all the steel tubes reached the yielding stress. Furthermore, the confinement pressure depends on a series of factors:

- The introduction of the load in the system.

If the load is applied to the composite section (steel+concrete) then the higher Poisson’s coefficient of the steel increases the value of its diameter faster than that of the concrete. As a consequence, the materials loose their contact\(^{19}\) until the concrete cracks and, with the increase of its apparent Poisson’s coefficient, they contact again and eventually the steel provides a lateral confinement. Also, in this case, the steel is subjected to a state of tension and compression and, according to Von Mises, the lateral stress has to be lower than $f_y$.

However, if all the load is introduced in the concrete, the steel provides confinement from the early stages, increasing as the concrete cracks and suffers a lateral expansion. This time, the maximum stress at the steel can in theory be up to $f_y$.

- The interface between the steel and the concrete. The influence of the interface can be regarded as the transference of forces between the two systems. For instance, in the case where load is introduced only in the concrete, a fraction of this load may be transfered due to the bond of the interface to the steel tube, reducing the confinement capacity of the steel as it was previously explained.

- The cracking state of the concrete. Depending on the cracking state of the concrete, the apparent Poisson’s coefficient changes, increasing for higher levels of cracking. Then,

---

\(^{19}\)This phenomenon is also helped by the shrinkage that the concrete may have suffered.
the development of the confinement strains, due to compatibility reasons, is highly influenced by the cracking state of the concrete (that in his turn, is also influenced by the confinement pressure)

- The failure mode of the concrete. Not only the cracking state of the concrete is important but also its failure mode. With a smeared cracking (low confinement level or no confinement) the contact between the concrete cylinder and the steel tube will be uniform in all the interface surface. However, if the crack is localized in a strip (figure 3.14 (b)) as it happens in steel encased sections due to the effect of the lateral confinement [16] the effect of the steel tube is not evident in the stress state of the encased concrete, mainly because the contact surface may not be uniform through the interface between the steel and the concrete.

In order to reproduce the behaviour of steel encased columns with the model for confined concrete, it is proposed to use the same law of development of confinement stresses that was previously described for fibre reinforced concrete, see figure 3.35 (b). The maximum lateral strain that is applied to the section is equal to that given by the yielding stress of the steel:

\[
\sigma_{\text{lat}} = \frac{2e_s f_y}{\phi_c}
\]

being \(e_s\) the thickness of the tube and \(\phi_c\) the diameter of the concrete cylinder. With this consideration, good results have been obtained\(^{20}\) for the experimental data obtained by Lahlou and Lachemi [32] as can be seen in figures 3.38, 3.39 and 3.40. In these figures it has to be noticed that the \(e = 6.4\) mm specimen of \(f_c = 50\) MPa has been calculated with a maximum confinement stress of \(0.80 f_y\). This result, however, seems to be in accordance with the experimental measures performed by the authors where not all the steel tubes reached their yielding stress (for high thicknesses).

The results obtained show that the model gives a pretty good estimation of the response of the element for normal strength concrete and admissible for high and ultra–high strength concrete. In any case, taking into consideration the number of simplifications that have been adopted in order to transform the problem of steel encased concrete to the problem of confined concrete, the model seems to give a reasonable prediction of its response.

### 3.4 Tension–compression behaviour

The case where a tension–compression loading pattern is applied to a concrete specimen can be related to the previous ideas developed for the tensile and compressive resistance. However, two different cases should be considered, one for lateral traction and longitudinal compression and other for lateral compression and longitudinal traction (figure 3.41).

\(^{20}\)In the calculus, it has been adopted:

- \(\varepsilon_p \in (0.003; 0.004)\)
- \(\sigma_{\text{pre}} \in (0.25; 0.50)\sigma_{\text{lat}}\)
- \(\sigma_{\text{pos}} \in (0.75; 0.50)\sigma_{\text{lat}}\)
3.4 Tension–compression behaviour

Concerning the triaxial state of stresses, little experimental research has been made on the tension–compression behaviour. Due to this reason, the results obtained in the bidimensional case will be studied and conveniently used for the confined response in this work.

The analogy can be understood with the help of figure 3.42. In it, the lateral stress of the concrete modifies the transverse state of stresses that controls the longitudinal response of concrete.

3.4.1 Influence of the transverse state of stresses in the longitudinal response of concrete
Figure 3.40: Comparison of the analytical model (continuous line) with the test results from Lahlou and Lachemi [32] (dashed line) for a concrete of $f_c = 130$ MPa for different thicknesses of the steel tube: (a) $e = 3.2$ mm; (b) $e = 6.4$ mm; (c) $e = 8.4$ mm.

(a) $e = 3.2$ mm $f_c = 130$ MPa (b) $e = 6.4$ mm $f_c = 130$ MPa (c) $e = 8.4$ mm $f_c = 130$ MPa

Figure 3.41: Loading cases considered: (a) compression failure; (b) tensile failure.

The element as it was previously explained.

Figure 3.42: Influence of the confinement pressure in the transverse state of stresses. Analogy between the 2 and 3 dimensional cases.

This influence has been studied in a bidimensional case for instance by Kupfer [30] and Hampel et alli [22] for different types of concrete ranging from 20 to 90 MPa. The different experimental results are presented in figure 3.43, where it is interesting to notice that the response of concrete is highly dependant on its strength.
In this work, it is proposed to adopt a simplified law consisting of a linear relationship between the two variables $\sigma_3$ and $\sigma_{\text{lat}}$ (see figure 3.43). The relationship can analytically be expressed as:

$$\frac{\sigma_3}{f_c} = 1 - \frac{\sigma_{\text{lat}}}{f_{cl}}$$

where $\sigma_{\text{lat}}$ can be identified as the confining pressure and $\sigma_3$ as the longitudinal stress. This idea is going to be exploited for the two possible cases, comprising lateral compression and tension in the following subsections.

### 3.4.2 Failure under longitudinal compression

If a certain lateral tension is applied to the concrete specimen, the transverse tension tie that controls the development of longitudinal cracking (figure 3.13) is pretensioned when the longitudinal load is applied. Due to this reason, the failure mechanism can in principle be assumed to be the one previously explained for a concrete without lateral confinement (figure 3.14 (a)) and a pure cohesive response is expected.

The failure starts when the lateral stress (due to the external tensile stress and the lateral tensile stress induced by the vertical loading) exceeds the tensile resistance of the concrete. In fact, this mechanism can be seen as a reduction of the transverse stress that the concrete cylinder can achieve, and thus the compressive resistance is reduced in a quantity that it is proposed to be estimated as\(^{21}\):

$$f_{cc} = f_c - f_c \frac{\sigma_{\text{lat}}}{f_{cl}} \approx f_c + 10\sigma_{\text{lat}}$$

Using this relationship, and taking into consideration that:

- $E_{cc} = E_c \frac{1}{1-2\nu_c}$ where $\lambda = \frac{\sigma_{\text{lat}}}{\sigma_3}$ and $\nu_c$ is the Poisson’s coefficient of the concrete
- A pure cohesive failure mode is expected (as it was seen for the concrete without confinement pressure)

\(^{21}\)Where the compressive resistance has a negative value and the lateral pressure a positive one.
It can be applied a modified TTJ diagram as follows:

\[ \sigma_3 = f_{cc} \left[ \frac{n \left( \frac{\varepsilon_3}{\varepsilon_c} \right)}{n - 1 + \left( \frac{\varepsilon_3}{\varepsilon_c} \right)^n k} \right] \]  

(3.39)

where \( n = 0.8 + \frac{f_c (\text{MPa})}{22} \); \( \varepsilon_c = \frac{f_{cc}}{E_{cc}} \frac{n}{n - 1} \) and \( k = 0.6 + \frac{f_{cc} (\text{MPa})}{50} \geq 1.0 \).

The results obtained applying this equation can be seen in figure 3.44 for different levels of lateral stress and assuming that \( f_c = 10 f_{ct} \) for two types of concrete. These results are not compared with test results due to the lack of experimental evidence on this field.

Figure 3.44: Results obtained with the analytical model for concretes with different compressive resistances: (a) \( f_c = 35 \text{ MPa} \); (b) \( f_c = 15 \text{ MPa} \).

From this figure it can be noticed that the energy under the \( \sigma - \varepsilon \) curve is not constant for different levels of \( \sigma_{lat} \). This fact is explained due to the external work introduced by the tensile lateral pressure in the specimen once the elements cracks longitudinally and suffers a lateral expansion.

Concerning the damage and the development of plastic strains, it is considered valid the considerations for the unconfined response that were previously stated in equation (3.31).

### 3.4.3 Failure under longitudinal tension

The second possible failure mode considered corresponds to the failure under tension when a lateral confinement pressure is applied.

Concerning the biaxial resistance of concrete under a bicompression state, experimental results show that it is increased around 10−25% depending on the concrete compressive uniaxial resistance\(^{22}\). As a result, considering a mean increment of 15% and assuming a linear increase in the longitudinal tensile stresses, it is obtained that:

\[ f_{cct} = f_{ct} - \frac{\sigma_{lat}}{1.15 f_c} f_{ct} \approx f_{ct} + 0.1 \sigma_{lat} \]  

(3.40)

This value \( f_{cct} \), the elastic modulus \( E_{cc} \) (obtained as in the previous subsection) and the fracture energy \( G_{Fc} \) are the ones that should be introduced in the tie model described in figure 3.45 in order to obtain the response of the element.

\(^{22}\)For instance, the experiments carried out by Curbach and Hampel at Dresden [12] for concrete ranging from 60 to 95 MPa, show that the higher the uniaxial compressive resistance, the lower the biaxial increase in the resistance.
For the latter parameter \((G_{Fc})\) it is also proposed to reduce its value proportional to the \(f_{ct}\) in such a way that the maximum crack width capable of transmitting stresses remains constant. According to Hordijk, \(w_c = 5.14\frac{G_F}{f_{ct}}\), and so:

\[
G_{Fc} = G_F \left( 1 + 0.1 \frac{\sigma_{lat}}{f_{ct}} \right) \tag{3.41}
\]

One example for the softening law with this consideration can be seen in figure 3.8.

Figure 3.45: Results obtained for the softening curve of the FPZ for a concrete of \(f_c = 35\) MPa with the analytical model
Chapter 4

Maintained loading

4.1 Introduction

This chapter is devoted to the study of the response of concrete under maintained loading as well as the development of a physical model in order to explain it. The model focuses on the unconfined case due to the scarce experimental data on the concrete’s confined long-term response but it will be extended to this case taking certain points into consideration.

4.2 Physical 1–D model

In this section a non-linear time–dependent model is presented in order to study the coupled behaviour of the viscous and plastic rheological strains of unconfined concrete.

4.2.1 Creep and shrinkage of concrete

The concrete exhibits a rheological behaviour developing delayed strains due to different processes whose origin is found in its microstructure [45, 19]. Conventionally, these strains are separated into shrinkage and creep, one (shrinkage) comprising the strains that appear if no external load is applied and other (creep) comprising the delayed strains due to the application of a external load (defined as the difference of the total delayed strains minus the shrinkage strains). Although this definition has certain inconsistencies\(^1\) it allows to represent the phenomenon in a simple way. For instance, the strains at a certain time for a concrete loaded at age \(t_0\) may be written as:

\[
\varepsilon(t, \frac{\sigma}{f_c}) = \varepsilon(t_0, \frac{\sigma}{f_c}) + \Delta \varepsilon_{cs}(t, t_0) + \Delta \varepsilon_{cc}(t, t_0, \frac{\sigma}{f_c})
\]  \(4.1\)

where the creep strains can be obtained from the others as:

\[
\Delta \varepsilon_{cc}(t, t_0, \frac{\sigma}{f_c}) = \varepsilon(t, \frac{\sigma}{f_c}) - \varepsilon(t_0, \frac{\sigma}{f_c}) - \Delta \varepsilon_{cs}(t, t_0)
\]  \(4.2\)

and its value can be expressed in the following way:

\[
\Delta \varepsilon_{cc}(t, t_0, \frac{\sigma}{f_c}) = \varepsilon(t_0, \frac{\sigma}{f_c}) \varphi(t, t_0, \frac{\sigma}{f_c})
\]  \(4.3\)

\(^1\)See for instance the “apparent mechanisms” defined by Wittmann [3].
being $\varphi(t, t_0, \frac{\sigma}{f_c})$ the creep coefficient of the concrete obtained from the rest of the measurable parameters.

By definition, the shrinkage strains are independent of the state of stresses of the concrete, and so, they represent a pure viscous behaviour of the material. However, the creep (although it exhibits a quasi–linear behaviour independent of the stresses for relationships up to $\sigma \leq 0.4f_c$) is directly related to the state of stresses and cracking of the concrete. Applying the definition of the creep strains (equation 4.2), different relationships have been studied to evaluate the non–linear effect of the stresses in the value of the creep coefficient [17]. These relationships correct the linear prediction of the creep coefficient introducing a parameter in the following way:

$$\Delta \varepsilon_{cc} = \varepsilon(t_0) \cdot \varphi(t, t_0, \frac{\sigma}{f_c}) = \varepsilon(t_0) \cdot \varphi_{lin}(t, t_0) \cdot \eta \left( \frac{\sigma}{f_c} \right)$$  \hspace{1cm} (4.4)

Where $\varphi_{lin}(t, t_0)$ is the value of the linear creep coefficient. According to Fernández, del Pozo and Arrieta [17] the value of $\eta \left( \frac{\sigma}{f_c} \right)$ can be estimated for the ascending branch of the stress–strain curve (for ratios $\sigma_c/f_c$ up to 0.7) as follows:

$$\eta \left( \frac{\sigma}{f_c} \right) = 1 + 2 \left( \frac{\sigma}{f_c} \right)^4$$  \hspace{1cm} (4.5)

The relationship between the creep strains and the stresses of the concrete can also be interpreted as a relationship between the creep strains and the cracking state of the concrete, showing them a coupled behaviour [14]. Due to this reason, it is proposed in this work that not all the delayed strains due to the creep of the concrete have a viscous nature but also a plastic one.

### 4.2.2 Effect of the delayed strains in the response of concrete

As it has been explained in the previous section, the delayed strains can be separated in principle into a viscous part and a plastic one, affecting in the reloading path of the material as can be seen in figure 4.1.

![Figure 4.1: Delayed strains caused by a maintained load](image)

According to this figure, the concrete is loaded to the point $A$ where it has a certain plastic strain $\varepsilon_{p,A}$. Once this point is reached, the load is maintained and delayed strains develop.
4.3 Plastic strains developed in a rheological process

Due to the creep and shrinkage of the concrete. Finally, when the rheological process finishes, the point has arrived to the position named $B$ where it can be reloaded to point $C$. At it, the total strain may be computed as:

$$\varepsilon_A = \varepsilon_{p,A} + \frac{\sigma}{E_A} \quad (4.6)$$

$$\varepsilon_B = \varepsilon_{p,A} + \Delta \varepsilon_v + \Delta \varepsilon_p + \frac{\sigma}{E_B} \quad (4.7)$$

In light of this, the total increase of the strain is equal to:

$$\Delta \varepsilon = \Delta \varepsilon_v + \Delta \varepsilon_p + \left( \frac{1}{E_B} - \frac{1}{E_A} \right) \sigma = \Delta \varepsilon_v + \Delta \varepsilon_p + \frac{(d_B - d_A)}{(1 - d_A)(1 - d_B)} \frac{\sigma}{E_0} \quad (4.8)$$

Where $d_A$ and $d_B$ are the damage parameters at points $A$ and $B$ respectively.

Concerning the development of the plastic strains, it has to be taken into account that if the increase of the plastic strains equals the maximum allowable for that level of stress, the concrete will crush, failing under static fatigue ($B \equiv C$), see figure 4.2.

![Figure 4.2: Static fatigue of the specimen under maintained load](image)

**4.3 Plastic strains developed in a rheological process**

Once the relationship between the plastic strains and the rheological behaviour has been posed, it remains the question of how much the plastic strains increase after a rheological process.

In this work, it is proposed that the linear part of the creep strains as well as the shrinkage strains correspond to the viscous part of the delayed strains, corresponding the rest (due to non–proportional creep) to anelastic strains. This relationship can be expressed as follows:

$$\Delta \varepsilon \left( t, t_0, \frac{\sigma}{f_c} \right) = \frac{\Delta \varepsilon_v}{\varepsilon(t_0) \varphi_{lin}(t_0) + \Delta \varepsilon_{cr}(t, t_s)} + \left( \frac{d_B - d_A}{(1 - d_A)(1 - d_B)} \frac{\sigma}{E_0} \right) \frac{\Delta \varepsilon_p}{\Delta \varepsilon_{cr}} \quad (4.9)$$

where, if the affinity hypothesis is accepted, it is obtained that $\Delta \varepsilon_{cr} = \varepsilon(t_0)(\eta - 1)\varphi_{lin}(t, t_0)$ and so,
\[ \Delta \varepsilon \left( t, t_0, \frac{\sigma}{f_c} \right) = \varepsilon(t_0) \varphi_{ln}(t, t_0) \eta + \Delta \varepsilon_{cs}(t, t_s) \] (4.10)

4.4 Behaviour under compression

This section is devoted to the verification of the previous hypothesis of the rheological behaviour and equation (4.9) with experimental results under compression.

4.4.1 Compression tests

Under compression, and in order to verify the non-linear creep behaviour and the development of plastic strains, it is studied the experimental results obtained by Rüsch [52] and Avram [2] presented in the so called Rüsch diagrams, see figure 4.3.

![Figure 4.3: Rüsch curves: (a) test results by Rüsch [52], taken from Denarié [14]; (b) test results by Avram [2]](image)

Two different domains can be distinguished in the curves. In the first one, for stresses lower than \(0.75 - 0.80 f_c\), the concrete does not fail under maintained load, whereas in the second one, for stresses greater than \(0.75 - 0.80 f_c\), the concrete crushes under maintained load (phenomenon called static fatigue).

For the creep limit branch, figure 4.4 shows a comparison of the proposed relationship with the experimental results under maintained load gathered in [17] for various authors and the results obtained by Denarié [14] for three concrete specimens\(^2\) tested under pure compressive relaxation conditions.

The static fatigue behaviour can be compared with the experimental tests performed by Rüsch and Avram\(^3\). The results, plotted in figure 4.5 show again good agreement at the creep limit and static fatigue failure.

It is remarkable the good agreement even using the affinity hypothesis on the static fatigue branch of the envelope, where the unstable crack growth may introduce certain differences in the shape of the development of strains with time.

\(^2\)Named M1194, M1196 and M1191 by the author.

\(^3\)For the monotonic loading behaviour, the constitutive diagram proposed in this work was used.
4.5 Behaviour under tension

Under tension, three different stages can be distinguished:

1. According to Neville [42], for stresses under $0.40f_{ct}$ and before the cracking process starts, it can be assumed a linear creep behaviour with a value of the creep coefficient similar to that of compression, thus, in this phase:

   $$\varphi_{lin,t}(t, t_0) = \varphi_{lin,c}(t, t_0)$$

2. For relationships $\frac{\sigma}{f_c} > 0.40$ and before the cracking process starts, the concrete loses its creep linearity. Some indirect measures of this phenomenon [19] show that in this phase the non-linear relationship proposed for the concrete under compression is also applicable:

   $$\varphi_t(t, t_0, \frac{\sigma}{f_c}) = \varphi_{lin,t}(t, t_0) \left[ 1 + 2 \left( \frac{\sigma}{f_c} \right)^4 \right] \rightarrow \eta = 1 + 2 \left( \frac{\sigma}{f_c} \right)^4$$

   In fact, this law can also be applied to describe the behaviour of the linear and non-linear phase.
3. After the peak load has been reached, and according to the theory developed in chapter 3, see figure 3.3, the stress is almost constant increasing only at a very slow rate over the effective (non–cracked) surface of the tie. This means that the level of non–linearity does not increase as much as before, although a little increment is expected. Due to this reason, for this phase it is proposed in this study the following relationship:

\[
\varphi_t(t, t_0, \frac{\sigma}{f_c}) = \varphi_{lin,t}(t, t_0) \left[ 2 + \frac{\sigma}{f_c} \right] \rightarrow \eta = 2 + \frac{\sigma}{f_c} \tag{4.13}
\]

In this document, verification of these relationships is proposed using the experimental data obtained by Denarié in [14]. These tests were conducted for concrete specimens subjected to different types of loading, mainly pure relaxation under compression and tension.

However, it should be highlighted that the relaxation steps performed by Denarié lasted approximately one hour. This means that the results are not given at time infinite and if the linear and non–linear creep curves present a non–proportional shape, differences in the value of \( \eta \) may appear.

### 4.5.1 Pure relaxation tensile tests

The first group of specimens tested by Denarié [14] were subjected to pure relaxation conditions. Different measures were taken, focusing on the displacement at the fracture zone and the displacement at the position where the forces were applied. In any case, the author found that both measures gave similar results and could be used to describe the relaxation behaviour of the specimen, see figure 4.6.

In order to study the non–linear creep coefficient of the concrete under tension (including also the post–peak branch) it is used the pure relaxation coefficient, defined as:

\[
R = \frac{F}{F_0} = 1 - \frac{\varphi}{1 + \chi \varphi} \tag{4.14}
\]

where \( \varphi \) is the non–linear creep coefficient \((\varphi = \eta \varphi_{lin})\) and \( \chi \) is the non–linear ageing coefficient. If the value of the relaxation coefficient is known, and using the values for the \( \chi \) coefficient proposed in [18] for the non–linear ageing coefficient, the creep coefficient can be obtained as:

\[
\varphi = \frac{1 - R}{1 - \chi (1 - R)} \tag{4.15}
\]

According to the acoustic measures performed during the tests (see figure 4.7) and except in the first and last relaxation steps, all the deformations that took place in the specimens were due to the creep of the concrete. This means that the first and last relaxation measures are not comparable to the rest because they would give “apparent” creep coefficients greater than the real ones (thanks to the fact that the total movement includes not only the creep strains but the opening due to the cracking of the specimen).

In the following, the results obtained by Denarié for different specimens will be used to obtain the value of the tensile creep coefficient in the following way:

1. The second relaxation measurement (with a ratio \( \frac{\sigma}{f_c} < 0.50 \)) will be adopted to calculate the linear creep coefficient
4.5 Behaviour under tension

2. Using equation (4.15) (and an appropriate value for the $\chi$ coefficient\footnote{In fact, due to the small value of the creep coefficient, the value of the ageing coefficient has little influence in the results.}, see [18]), the creep coefficient will be obtained from the relaxation coefficient ($R$).

3. Dividing the creep coefficient between the linear creep coefficient, the value of the parameter $\eta$ will be derived as it was done in the compression case ($\eta = \frac{\chi}{\phi \phi_{lin}}$).

The results obtained applying this methodology are shown in figure 4.8 where the proposed analytical relationship is also plotted for the different specimens.

4.5.2 Pure tension tests

Some tests were also carried by Denarié [14] for wedge specimens loaded under pure tension. For them, and not considering again the first loading step due to the development of cracking, the creep coefficient can be obtained using the relationship:
Figure 4.7: Acoustic measurements during the tests [14]

Figure 4.8: Comparison of test results from Denarié [14] with the analytical model: (a) M11914 specimen; (b) M11916 specimen; (c) M12026 specimen

$$\varphi = \frac{w(t)}{w(t_0)} - 1$$  \hspace{1cm} (4.16)

The results obtained can be seen in figure 4.9.

### 4.6 Conclusions

The comparison of the theoretical model with the experimental results shows good agreement under tension and compression for all ranges of stresses\(^5\).

Under compression, only pre-peak experimental results have been found, but they seem to be in good accordance even for high level of stresses in the concrete (static failure cases).

\(^5\)Although a greater dispersion has been found for higher $\frac{f}{f_c}$ relationships.
4.7 Confined response

Under tension, pre and post–peak experimental results have been used in order to validate the model obtaining again good agreement and defining the following stages of behaviour:

1. A first stage with stresses up to 0.40 \( f_c \), where the value of the \( \eta \) coefficient is similar to 1.0 (linear creep)

2. A second stage up to the peak strain, where a clear non–linear behaviour is shown, following a relationship close to \( \eta = 1 + 2 \left( \frac{\sigma}{f_c} \right)^4 \)

3. A third stage, after the peak stress, where the non–linear behaviour is smoothed thanks to the fact that both the external force and the effective surface of concrete decrease and, as a consequence, the level of stress is maintained or slightly increased. In this stage, the adopted relationship has been: \( \eta = 2 + \left( \frac{\sigma}{f_c} \right) \)

4.7 Confined response

Concerning the rheological triaxial response of concrete, there is a certain agreement between authors [5, 40] in accepting the validity of the superposition principle. Furthermore, Naguib and Mirmiran [40] accept this principle “despite anisotropy and creep nonlinearity” based on the experimental results of [28, 20]. According to these authors, for the confined concrete, the axial creep strain \( \varepsilon_{c,a} \) can be estimated as:

\[
\varepsilon_{c,a} = \varepsilon_{c,a,uniax} - 2\nu_c J(t, t_0)f_r
\]

(4.17)

where \( f_r \) is the confinement pressure, \( J(t, t_0) \) the compliance function and \( \nu_c \) the short term static Poisson’s ratio for biaxial or triaxial loading (ranging between 0.09 to 0.17).

This behaviour can also be explained (for tension and compression) by means of a lattice model with elements implementing rheological behaviour if a cubic mesh is chosen\(^7\). Furthermore, according to Bažant [4], the creep coefficient under shear stresses has been found to

\(^6\) \( J(t, t_0) = 1 + \frac{\sigma(t,t_0)}{E(t_0)} \)

\(^7\)Whatever orientation is chosen, for the shrinkage strains it can always be assumed to be valid the principle of superposition.
be approximately equal to that under normal stresses ($\varphi_{\tau} \approx \varphi_{\sigma}$). This result can also be obtained assuming an uncoupled response in the two main directions of the stress tensor. 

As a consequence, and based on the previous considerations, the confined time-dependent behaviour of concrete under triaxial state of stresses will be assumed uncoupled in its different principal directions, accepting equation (4.17) as a valid formula to obtain the creep strains. In order to evaluate the non-linear creep coefficient ($\eta$) it is proposed to generalise its expression as follows:

$$\eta = 1 + 2 \left( \frac{\sigma_3}{f_{cc}} \right)^4$$

(4.18)

where $f_{cc} = f_c + 4\sigma_{lat}$. The reason for doing so is that $\eta$ depends mainly of the cracking state of the concrete and, for confined and unconfined specimens, it has been seen that around $0.70f_{cc} - 0.80f_{cc}$ the material loses its linearity due to the development of microcracking and accelerating the creeping process.

Concerning the static fatigue of concrete, it is expected according to the model an improved resistance of the material to maintained loading. This phenomenon is due to the change of the shape of the descending branch of the concrete, as can be seen in the short term diagrams of figure 4.10. To the limit, it can also be obtained that, in theory, for a triaxial loading state the concrete will never crush under static fatigue.

![Figure 4.10: Effect of the confinement stresses in the static fatigue resistance of concrete for the same level of stresses $\frac{\sigma_3}{f_{cc}}$](image-url)
Chapter 5

Cyclic loading

5.1 Introduction

The cyclic loading and its fatigue failure mode will be the second type of long–term loading studied. Its effects will also be treated in the framework of the coupled plasticity–damage model. As in the previous chapter, the model will be derived for the unconfined case due to the scarce experimental data on cyclic confined response and after it will extended to the latter making certain considerations.

5.2 Unconfined physical model

The physical model proposed to explain the fatigue behaviour of the concrete is based on the assumption that all its delayed strains (strains developed with the number of cycles) have an anelastic nature. This idea is presented in figure 5.1.

![Figure 5.1: Strain development for a concrete subjected to cyclic loading](image)

This idea can be analytically formulated as follows:

\[ \Delta \varepsilon_N = \Delta \varepsilon_p + \frac{(d_B - d_A)}{(1 - d_A)(1 - d_B)} \frac{\sigma}{E_0} \]  

(5.1)

In order to apply the model, the calculation of the increase of the total strain with the number
Cyclic loading of cycles ($\Delta \varepsilon_N$) is performed using the information given by the $S - N$ curves, the $\Delta \varepsilon - N$ relationship (see figure 5.2) and a damage accumulation criterion.

5.2.1 $S - N$ curves

The first input of the model, the $S - N$ curves, define the number of cycles of constant amplitude that the concrete is capable of bearing. They have been widely studied in the literature of the theme and exist these type of diagrams for the concrete under compression, tension or tension–compression. For instance, figure 5.3 shows the relationships adopted by Schläfli [53] and taken from the SIA D0133 norm (1997) for compression as well as the Cornelissen & Siemens curves (1985) for tension and tension–compression [11].

5.2.2 $\Delta \varepsilon - N$ relationship

In order to determine the total strain suffered by the concrete when a certain number of cycles have taken place, it is necessary also to take into consideration the $\Delta \varepsilon - N$ relationship for the material. This law allows to determine the increase of the strains for a given cycle taking into consideration the previous number of cycles.

For this purpose, the results obtained by Pfanner et alli [46] (after a probabilistic study of the evolution of the damage in concrete based on the test results by Holmen [24]) are of great
5.2 Unconfined physical model

Figure 5.3: $S - N$ design curves taken from [53]: (a) Compression, SIA D0133 (1997); (b) Tension and tension–compression, Cornelissen and Siemens (1985) [11].

interest. Figure 5.4 shows the curve proposed by Pfanner in order to calculate the evolution of the strains in the concrete as a certain number of cycles are applied to the concrete.

Figure 5.4: Analytical model of Pfanner et alli [46]: (a) definition of the different parameters; (b) proposed $\Delta\varepsilon - N$ curve based on a probabilistic study.

In reference [46], the authors detail a model for the $\Delta\varepsilon - N$ curve considering three different regions each one with a different law of behaviour. Although good agreement is found by the authors, in this study a continuous function is proposed in order to have an easier expression towards its implementation as follows:

$$
\varepsilon = \varepsilon_0 + \Delta\varepsilon \left[ \frac{1}{2} + \frac{\zeta}{((2\zeta)^2 - 2)^2} - \frac{\zeta}{((2\zeta)^2 - 2)} \right]
$$

(5.2)

which can also be expressed as:

$$
\varepsilon = \varepsilon_0 + \frac{\Delta\varepsilon}{2} \left[ 1 + \zeta \frac{3 - (2\zeta)^2}{((2\zeta)^2 - 2)^2} \right]
$$

(5.3)

where $\Delta\varepsilon = (\varepsilon_f - \varepsilon_0)$ and $\zeta$ is a parameter whose value is $\zeta = \frac{n}{N_F} - 0.5$ (being $n$ the number of cycles applied and $N_F$ the total number of cycles to produce the failure of the concrete).
A comparison of this law with the results found by Pfanner et alli can be seen in figure 5.5.

![Figure 5.5: Comparison of the results by Pfanner et alli [46] and the proposed relationship: (a) upper and lower fractiles based on Holmen tests [46]; (b) proposed relationship.](image)

The validity of this law has also been contrasted with the experimental tests carried out by Qingbin et alli [47], whose results can be seen in figure 5.6 for two levels of maximal load ($\sigma_{\text{max}} = 0.70f_c$ and $\sigma_{\text{max}} = 0.85f_c$). Again, good agreement is found with the analytical expression for the different tests performed.

![Figure 5.6: Comparison of the results by Qingbin et alli [47] and the proposed relationship (dashed line for experimental results and continuous line for analytical model): (a) results for $\sigma_{\text{max}} = 0.70f_c$; (b) results for $\sigma_{\text{max}} = 0.85f_c$.](image)

### 5.2.3 Accumulation of damage

Finally, a criterion for the accumulation of damage in concrete is required to study the fatigue effect of variable amplitude cycles. According to Schläfli [53], the Palgrem–Miner criterion for damage accumulation does not seem to provide good results and no definitive answer has been given to this question.

In this work, it is proposed to study the accumulation of damage with the previously seen model of plastic strains. Then, for a given concrete under a certain state of fatigue loading, this fatigue loading state is defined by $\sigma_{\text{max},1}$ and $\Delta \sigma_1$. With these two values, it can be estimated $N_{F,1}$ (the maximum number of cycles resisted under cyclic loading) from the $S - N$ curves.
it will be obtained that after a determined number of cycles, the anelastic strains will have increased in a certain quantity. If later the same concrete is reloaded with other fatigue loading pattern (different $\sigma_{\text{max},2}$ and $\Delta\sigma_2$ in principle) the previous history of anelastic strains will be reflected as an equivalent initial number of cycles; this idea is shown in figure 5.7.

![Diagram](image)

**Figure 5.7:** Accumulation of damage under different fatigue loading patterns

With this scheme, derived from the previous considerations, it is clear that the Palgrem–Miner linear criterion does not satisfy the damage accumulation for concrete because the previous history of fatigue loading is not taken into account.

The calculation sequence according to the previously described method may be sketched as follows:

1. Compute the initial state of stresses, deformations and damage
2. Calculate for the first fatigue loading the maximum number of cycles $N_{F,1}$ and the $\Delta\varepsilon - N$ relationship.
3. Calculate the anelastic strains for the number of cycles and compute the increase in the damage parameter
4. Unload or reload to the next level of fatigue loading and compute the initial equivalent anelastic strain for the new $\Delta\varepsilon - N$ curve.
5. Repeat the operations until the failure is reached or the load is stopped

In this process, when updating the anelastic strains, an implicit equation has to be solved because it is required the value of \( n \) for a given \( (\varepsilon - \varepsilon_0) \) in expression (5.3). However, due to the nature of the problem, it is solved in a fast and effective way with the string method if the limit points are chosen as the initial test values, see figure 5.8.

![Application of the string method to solve the implicit equation](image)

Figure 5.8: Application of the string method to solve the implicit equation

### 5.3 Comparison with experimental results

As experimental contrast of the above described model, several experiences can be considered. For instance, for concrete under tension, figure 5.9 represents a experimental experience of Alleruzo [1] performed in 1997. It corresponds to a concrete subjected to cyclic loading under tension in the post–peak branch of the curve. Over the original picture, it has been added what can be considered as the static resistance of the specimen as well as the stiffness moduli for the initial and final situations. The previously commented points of the increase of the plastic strains can clearly be seen as well as the validity of the coupled plasticity–damage model to reproduce the observed behaviour.

![Fatigue tension behaviour](image)

Figure 5.9: Fatigue tension behaviour, Alleruzo [1] (1997), taken from [53]

Also, for compression, the model seems to provide pretty good agreement with experimental
evidences. The coupled plasticity–damage response of concrete has also been proposed by many authors when dealing with the fatigue behaviour of concrete. For instance, this idea has been introduced by some authors considering the focal point model (see for instance Park [44], Rotilio [51], Schläfli [53], ...) when studying cyclic loading, as can be seen in figure 5.10. This last figure can also be considered for the damage accumulation criterion that has been previously presented.

![Figure 5.10: Compression model for concrete (Park [44], 1990, taken from Schläfli [53])](image)

Based on the experiences of Holmen, El-Tawil et alii [15] developed a constitutive model for concrete subjected to fatigue loading where again the development of the plastic strains was coupled with the damage in the concrete as can be seen in figure 5.11.

![Figure 5.11: Compression model for concrete (El-Tawil et alii [15], 2001)](image)
5.4 Confined response

Once the cyclic unconfined response of concrete has been defined, its extension to the confined situation will be studied in this section.

Concerning this theme, there is a great lack of experimental data for the different inputs of the model, the $S - N$ curve and $\Delta \varepsilon - N$ relationship. As in the case of the rheological strains, according to the model it is expected a much higher resistance to the fatigue loading if a confinement pressure is applied to the concrete. The main reason can be found in the fact that the shape of the $\sigma - \varepsilon$ curve changes, see figure 5.12, and it can even disappear its descending branch. If this happens, the fatigue failure can not appear because the development of plastic strains will not intercept the descending branch of the curve. In general, it can then be affirmed that it is expected that the confinement increases notably the fatigue resistance of the concrete.

![Figure 5.12: Effect of the confinement stresses in the fatigue resistance of a concrete specimen](image)

This phenomenon has also an influence on the development of plastic strains. In this theme, and regarding the unloading–reloading cycles for confined concrete (figure 3.31) it can be seen that certain plastic strains are developed in each loading loop and an hysteretic energy is wasted in the process. Also, some indirect measures of the cyclic response of confined concrete (like for instance in bond–slip problems of reinforced concrete) show that the concrete increases its strains with the number of cycles.

Due to these reasons it is proposed in this work to consider that although the number of cycles that the concrete can bear under cyclic loading is augmented with increasing confinement, the rate of development of plastic strains is maintained for a level of stress. Then, if there is a confined concrete with a ratio $\sigma / f_c = K$, its response can be related to that of unconfined concrete with $\sigma / f_c = K$ in the following way:

$$\frac{d\varepsilon}{dn}\bigg|_U = \frac{d\varepsilon}{dn}\bigg|_C$$

(5.4)

Where the subscript $U$ means unconfined response and $C$ confined. This equation can be used to obtain the maximum number of cycles that the confined concrete can bear ($N_{F,c}$) if the same value is known for the unconfined case ($N_{F,u}$) taking equation (5.3) into consideration:

$$\varepsilon = \varepsilon_0 + \Delta \varepsilon_T [f(\zeta)]$$

(5.5)
and deriving,

$$\frac{d\varepsilon}{d\zeta} = \Delta\varepsilon_T [f'(\zeta)]$$

(5.6)

but considering that:

$$\zeta = \frac{n}{N_F} - 0.5 \rightarrow d\zeta = \frac{dn}{N_F}$$

(5.7)

that can be introduced in the previous equation, giving:

$$\frac{d\varepsilon}{dn} = \frac{1}{N_F} \Delta\varepsilon_T [f'(\zeta)]$$

(5.8)

finally, applying the proposed relationship (5.4) for the confined and unconfined case, it is obtained:

$$\frac{\Delta\varepsilon_{T,u}}{N_{F,u}} = \frac{\Delta\varepsilon_{T,c}}{N_{F,c}} \rightarrow N_{F,c} = N_{F,u} \frac{\Delta\varepsilon_{T,c}}{\Delta\varepsilon_{T,u}}$$

(5.9)

For the case where the descending branch of the curve is not intercepted, a conventional limit strain can be adopted for the concrete, for instance $\Delta\varepsilon_{T,c} = 10\%$, and the rest of the calculus can be performed as it has been detailed.
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